Vector-valued polynomials and a matrix weight function with B_2-action II

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ABSTRACT. This is a sequel to SIGMA 9 (2013), 007, 23 pp. (arXiv:1210.1177), in which there is a construction of a 2×2 positive-definite matrix function $K(x)$ on $\mathbb{R}^2$. The entries of $K(x)$ are expressed in terms of hypergeometric functions. This matrix is used in the formula for a Gaussian inner product related to the standard module of the rational Cherednik algebra for the group $W(B_2)$ (symmetry group of the square) associated to the (2-dimensional) reflection representation. The algebra has two parameters: k_0, k_1. In the previous paper K is determined up to a scalar, namely, the normalization constant. The conjecture stated there is proven in this note. An asymptotic formula for a sum of $_3F_2$-type is derived and used for the proof.

1. Introduction

This is a sequel to [1] and the definitions and notations from that paper are used here. Briefly, we constructed a 2×2 positive-definite matrix function $K(x)$ on $\mathbb{R}^2$ whose entries are expressed in terms of hypergeometric functions. This matrix is used in the formula for a Gaussian inner product related to the standard module of the rational Cherednik algebra for the group $W(B_2)$ (symmetry group of the square) associated to the (2-dimensional) reflection representation. The algebra has two parameters: k_0, k_1. In [1] K is determined up to a scalar, namely, the normalization constant. The conjecture stated there is proven in this note.

Instead of trying to integrate K directly (a problem involving squares of hypergeometric functions whose argument is x_2^2/x_1^2) we compute a sequence of integrals in two ways: asymptotically and exactly in terms of sums. Comparing the two answers will determine the value of the normalizing constant. Recall from [1, p.18]

that for $0 < x_2 < x_1$ and $u = \frac{x_2}{x_1}$

(1.1) $L\left(u\right)_{11} = |u|^{k_1}\left(1-u^2\right)^{-k_0} F\left(-k_0, \frac{1}{2} - k_0 + k_1; k_1 + \frac{1}{2}; u^2\right)$

$L\left(u\right)_{12} = -\frac{k_0}{k_1 + \frac{1}{2}}|u|^{k_1}\left(1-u^2\right)^{-k_0} uF\left(1 - k_0, \frac{1}{2} - k_0 + k_1; k_1 + \frac{3}{2}; u^2\right)$

$L\left(u\right)_{21} = -\frac{k_0}{\frac{1}{2} - k_1}|u|^{-k_1}\left(1-u^2\right)^{-k_0} uF\left(1 - k_0, \frac{1}{2} - k_0 - k_1; \frac{3}{2} - k_1; u^2\right)$

$L\left(u\right)_{22} = |u|^{-k_1}\left(1-u^2\right)^{-k_0} F\left(-k_0, \frac{1}{2} - k_0 - k_1; \frac{1}{2} - k_1; u^2\right),$

and ([**1**, p.20])

(1.2) $K\left(x\right) = L\left(u\right)^T \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} L\left(u\right),$

$d_1 = c\frac{\Gamma\left(\frac{1}{2} - k_1\right)^2}{\cos\pi k_0 \Gamma\left(\frac{1}{2} + k_0 - k_1\right)\Gamma\left(\frac{1}{2} - k_0 - k_1\right)},$

$d_2 = c\frac{\Gamma\left(\frac{1}{2} + k_1\right)^2}{\cos\pi k_0 \Gamma\left(\frac{1}{2} + k_0 + k_1\right)\Gamma\left(\frac{1}{2} - k_0 + k_1\right)}.$

2. Integrals over the circle

First we reduce the integral to a sector of the unit circle by assuming homogeneity and an invariance property. As before, elements of $\mathcal{P}_V$ can be expressed as polynomials $f_1\left(x\right)t_1 + f_2\left(x\right)t_2$ or vectors $\left(f_1\left(x\right), f_2\left(x\right)\right)$, as needed. The Gaussian inner product is expressed as

$$\langle f, g\rangle_G = \int_{\mathbb{R}^2} f\left(x\right)K\left(x\right)g\left(x\right)^T e^{-|x|^2/2}dx,$$

and the normalization condition is equivalent to $\langle(1,0),(1,0)\rangle_G = 1$. Let $x_\theta := (\cos\theta, \sin\theta)$, a generic point on the unit circle.

LEMMA 1. *Suppose $f, g \in \mathcal{P}_V$ are relative invariants of the same type, that is, for some linear character χ of W, $\left(wf\right)\left(x\right) = f\left(xw\right)w^{-1} = \chi\left(w\right)f\left(x\right)$ and $\left(wg\right)\left(x\right) = \chi\left(w\right)g\left(x\right)$ for each $w \in W$, then*

$$\int_{\mathbb{R}^2} f\left(x\right)K\left(x\right)g\left(x\right)^T e^{-|x|^2/2}dx = 8\int_0^\infty e^{-r^2/2}rdr\int_0^{\pi/4} f\left(rx_\theta\right)K\left(x_\theta\right)g\left(rx_\theta\right)^T d\theta.$$

PROOF. Let $C_0 = \{x : 0 < x_2 < x_1\}$, the fundamental chamber, then

$$\int_{\mathbb{R}^2} f\left(x\right)K\left(x\right)g\left(x\right)^T e^{-|x|^2/2}dx = \sum_{w\in W}\int_{C_0} f\left(xw\right)K\left(xw\right)g\left(xw\right)^T e^{-|x|^2/2}dx$$

$$= \sum_{w\in W}\int_{C_0} f\left(xw\right)w^{-1}K\left(x\right)wg\left(xw\right)^T e^{-|x|^2/2}dx$$

$$= \sum_{w\in W}\chi\left(w\right)^2\int_{C_0} f\left(x\right)K\left(x\right)g\left(x\right)^T e^{-|x|^2/2}dx.$$

The statement follows from the fact $\chi\left(w\right)^2 = 1$ and the use of polar coordinates. Recall K is positively homogeneous of degree zero.   □

Put this formula in the context of an abstract inner product over the circle (S^1) with the following:

DEFINITION 1. *For polynomials $f \in \mathcal{P}_{V,n}, g \in \mathcal{P}_{V,m}$ let $\ell = \frac{m+n}{2}$ and*

$$\langle f, g \rangle_S := \frac{1}{2^\ell \ell!} \langle f, g \rangle_G = \frac{1}{2^\ell \ell!} \left\langle e^{\Delta_\kappa/2} f, e^{\Delta_\kappa/2} g \right\rangle_\tau, \quad m \equiv n \bmod 2,$$

$$\langle f, g \rangle_S := 0, \quad m - n \equiv 1 \bmod 2.$$

This is extended to all polynomials by linearity.

PROPOSITION 1. *If $f, g \in \mathcal{P}_V$ are relative invariants of the same type and $f \in \mathcal{P}_{V,n}, g \in \mathcal{P}_{V,m}$ with $m \equiv n \bmod 2$ and $\ell = \frac{m+n}{2}$ then*

$$8 \int_0^{\pi/4} f(x_\theta) K(x_\theta) g(x_\theta)^T d\theta = \frac{1}{2^\ell \ell!} \langle f, g \rangle_G = \langle f, g \rangle_S.$$

If further $n = 2q + 1$ and $m = 1$ then

$$\langle f, g \rangle_S = \frac{1}{2^{2q+1} q! (q+1)!} \langle \Delta_\kappa^q f, g \rangle_\tau.$$

PROOF. From Lemma 1 the factor relating $\langle \cdot, \cdot \rangle_S$ to $\langle \cdot, \cdot \rangle_G$ is $\int_0^\infty e^{-r^2/2} r^{2\ell+1} dr = 2^\ell \ell!$. Now suppose $m = 1$ and $\ell = q+1$. By definition $\langle f, g \rangle_S = \dfrac{\left\langle e^{\Delta_\kappa/2} f, e^{\Delta_\kappa/2} g \right\rangle_\tau}{2^{q+1}(q+1)!}$, then $e^{\Delta_\kappa/2} g = g$ and the degree-1 component of $e^{\Delta_\kappa/2} f$ is $\dfrac{1}{2^q q!} \Delta_\kappa^q f$ (recall that $\langle f, g \rangle_\tau = 0$ when f and g have different degrees of homogeneity). $\quad\square$

We will find exact expressions in the form of sums for $\Delta_\kappa^q f$ for certain polynomials.

The idea underlying the asymptotic evaluation is this: suppose that g is continuous on $[0, 1]$ and satisfies $|g(t) - g(0)| \le Ct$ for some constant, then $\int_0^1 g(t)(1-t)^n dt = \frac{1}{n+1} g(0) + O\left(\frac{1}{n^2}\right)$. This formula can be adapted to the measure $t^\alpha dt$ with $\alpha > -1$.

Let $\phi := x_1^2 - x_2^2$; then ϕ^2 is W-invariant. Furthermore $\phi^{2n} p_{1,2}$ and $\phi^{2n+1} p_{1,4}$ are all relative invariants of the same type, that is, $\sigma_1 f = \sigma_{12}^+ f = -f$ (recall $p_{1,2} = -x_2 t_1 + x_1 t_2$ and $p_{1,4} = -x_2 t_1 - x_1 t_2$). We will evaluate $\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S$ and $\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S$. These polynomials peak at $\theta = 0$ and vanish at $\theta = \frac{\pi}{4}$. The following are used in the expressions for $p_{1,4} K p_{1,2}^T$ and $p_{1,2} K p_{1,2}^T$. Set

$$(2.1) \qquad h_1(z) := F\left(-k_0, \frac{1}{2} - k_0 + k_1; \frac{3}{2} + k_1; z\right),$$

$$h_2(z) := F\left(-k_0, -\frac{1}{2} - k_0 - k_1; \frac{1}{2} - k_1; z\right),$$

$$h_3(z) := F\left(k_0, \frac{1}{2} + k_0 + k_1; \frac{3}{2} + k_1; z\right),$$

$$h_4(z) := F\left(k_0, -\frac{1}{2} + k_0 - k_1; \frac{1}{2} - k_1; z\right).$$

Each of these hypergeometric series satisfies the criterion for absolute convergence at $z = 1$ (for real $F(a, b; c; z)$ the condition is $c - a - b > 0$; here $c - a - b = 1 \pm 2k_0$),

and so each satisfies a bound of the form $|h(z) - 1| \leq Cz$ for $0 \leq z \leq 1$. Recall the coordinate $u = \frac{x_2}{x_1}$; on the unit circle (in $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$)

$$x_\theta = \left(\frac{1}{\sqrt{1+u^2}}, \frac{u}{\sqrt{1+u^2}} \right), \quad \phi = \frac{1-u^2}{1+u^2}, \quad d\theta = \frac{du}{1+u^2}.$$

By use of the identities

$$(2.2) \qquad F(a,b;c;z) - \frac{a}{c} z F(a+1,b;c+1,z) = F(a,b-1;c;z),$$

$$F(a,b;c;z) - \frac{a}{c} F(a+1,b;c+1;z) = \frac{c-a}{c} F(a,b;c+1;z),$$

we obtain for $x = x_\theta, 0 < \theta < \frac{\pi}{4}$ and $0 < u < 1$:

$$(2.3) \qquad x_2 L_{11} - x_1 L_{12} = \frac{u^{k_1+1} \left(1-u^2\right)^{-k_0}}{\left(1+u^2\right)^{1/2}} \frac{1+2k_0+2k_1}{1+2k_1} h_1\left(u^2\right),$$

$$x_1 L_{22} - x_2 L_{21} = \frac{u^{-k_1} \left(1-u^2\right)^{-k_0}}{\left(1+u^2\right)^{1/2}} h_2\left(u^2\right);$$

and

$$(2.4) \qquad -x_2 L_{11} - x_1 L_{12} = -\frac{u^{k_1+1} \left(1-u^2\right)^{k_0}}{\left(1+u^2\right)^{1/2}} \frac{1-2k_0+2k_1}{1+2k_1} h_3\left(u^2\right),$$

$$-x_2 L_{21} - x_1 L_{22} = -\frac{u^{-k_1} \left(1-u^2\right)^{k_0}}{\left(1+u^2\right)^{1/2}} h_4\left(u^2\right).$$

The expressions for L_{ij} (see 1.1) appearing in (2.4) are first transformed with $F(a,b;c;z) = (1-z)^{c-a-b} F(c-a,c-b;c;z)$ before using identities (2.2). These formulae will be used to obtain asymptotic expressions for the integrals $\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S$ and $\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S$. The notation $a(n) \sim b(n)$ means $\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1$.

LEMMA 2. *Suppose* $\alpha, \gamma > -1$, *and* $n = 2, 3, \ldots$ *then*

$$\int_0^1 t^\alpha (1-t)^{n+\gamma} (1+t)^{\beta-n} \, dt = (2n)^{-\alpha-1} \Gamma(\alpha+1) \left(1 + O\left(\frac{1}{n}\right)\right).$$

PROOF. Change the variable to $t = \frac{v}{2-v}$ then the integral becomes

$$2^{-\alpha-1} \int_0^1 v^\alpha (1-v)^{n+\gamma} \left(1 - \frac{v}{2}\right)^{-\alpha-\beta-\gamma-2} dv = 2^{-\alpha-1} \frac{\Gamma(\alpha+1)\Gamma(n+\gamma+1)}{\Gamma(n+\alpha+\gamma+2)} + R,$$

where R is bounded by $2^{-\alpha-1} C \int_0^1 v^{\alpha+1} (1-v)^{n+\gamma} \, dv$ and $\left| \left(1 - \frac{v}{2}\right)^{-\alpha-\beta-\gamma-2} - 1 \right| \leq Cv$ for $0 \leq v \leq 1$. By Stirling's formula $\frac{\Gamma(n+\gamma+1)}{\Gamma(n+\alpha+\gamma+2)} \sim n^{-\alpha-1}$ and $R \sim C'n^{-\alpha-2}$ for some constant C'. $\qquad \square$

COROLLARY 1. *Suppose* $g(t)$ *is continuous and* $|g(t) - g(0)| \leq Ct$ *for* $0 \leq t \leq 1$ *then*

$$\int_0^1 t^\alpha (1-t)^{n+\gamma} (1+t)^{\beta-n} g(t) \, dt = (2n)^{-\alpha-1} g(0) \Gamma(\alpha+1) \left(1 + O\left(\frac{1}{n}\right)\right).$$

PROOF. Break up the integrand as

$$t^\alpha (1-t)^{n+\gamma} (1+t)^{\beta-n} \{g(0) + (g(t) - g(0))\};$$

by the Lemma the integral of the second part is bounded by $C (2n)^{-\alpha-2} \Gamma(\alpha+2)$ $\left(1 + O\left(\frac{1}{n}\right)\right)$. $\qquad\square$

PROPOSITION 2. *Suppose* $-\frac{1}{2} < k_0 \pm k_1 < \frac{1}{2}$ *then*

$$\langle \phi^{2n} p_{1,2}, p_{1,2} \rangle_S = 8 \int_0^{\pi/4} \phi(x_\theta)^{2n} p_{1,2}(x_\theta) K(x_\theta) p_{1,2}(x_\theta)^T d\theta$$

$$\sim \frac{2\pi c}{\cos \pi k_0 \cos \pi k_1} \frac{2^{2k_1} n^{k_1-1/2} \Gamma\left(\frac{1}{2} + k_1\right)}{\Gamma\left(\frac{1}{2} + k_0 + k_1\right) \Gamma\left(\frac{1}{2} - k_0 + k_1\right)}.$$

PROOF. By definition

$$\phi(x_\theta)^{2n} p_{1,2}(x_\theta) K(x_\theta) p_{1,2}(x_\theta)^T = \left(x_1^2 - x_2^2\right)^{2n}$$

$$\times \left\{ d_1 \left(x_1 L_{12} - x_2 L_{11}\right)^2 + d_2 \left(x_1 L_{22} - x_2 L_{21}\right)^2 \right\}.$$

Thus equation (2.3) implies

$$8 \int_0^{\pi/4} \phi(x_\theta)^{2n} p_{1,2}(x_\theta) K(x_\theta) p_{1,2}(x_\theta)^T d\theta$$

$$= 8d_1 \left(\frac{1 + 2k_0 + 2k_1}{1 + 2k_1}\right)^2 \int_0^1 \left(\frac{1 - u^2}{1 + u^2}\right)^{2n} \frac{\left(1 - u^2\right)^{-2k_0}}{\left(1 + u^2\right)^2} u^{2k_1+2} h_1\left(u^2\right)^2 du$$

$$+ 8d_2 \int_0^1 \left(\frac{1 - u^2}{1 + u^2}\right)^{2n} \frac{\left(1 - u^2\right)^{-2k_0}}{\left(1 + u^2\right)^2} u^{-2k_1} h_2\left(u^2\right)^2 du,$$

where h_1 and h_2 are from equation (2.1). The key fact is that $h_i(0) = 1$ and $\left|h_i\left(u^2\right) - 1\right| \le Cu^2$ for $0 \le u \le 1$ with some constant C, $i = 1, 2$. In each integral change the variable $u = v^{1/2}$; the first integral equals

(2.5) $$\frac{1}{2} (4n)^{-k_1-3/2} \Gamma\left(k_1 + \frac{3}{2}\right) \left(1 + O\left(\frac{1}{n}\right)\right)$$

and the second integral equals

(2.6) $$\frac{1}{2} (4n)^{k_1-1/2} \Gamma\left(\frac{1}{2} - k_1\right) \left(1 + O\left(\frac{1}{n}\right)\right).$$

This is the dominant term in the sum because $k_1 - \frac{1}{2} > -k_1 - \frac{3}{2}$ (that is, $2k_1 + 1 > 0$). Using the value of d_2 in equation (1.2) and the identity $\Gamma\left(\frac{1}{2} - k_1\right) \Gamma\left(\frac{1}{2} + k_1\right) = \frac{\pi}{\cos \pi k_1}$ we find

$$\langle \phi^{2n} p_{1,2}, p_{1,2} \rangle_S \sim 4c (4n)^{k_1-1/2} \frac{\Gamma\left(\frac{1}{2} - k_1\right) \Gamma\left(\frac{1}{2} + k_1\right)^2}{\cos \pi k_0 \Gamma\left(\frac{1}{2} + k_0 + k_1\right) \Gamma\left(\frac{1}{2} - k_0 + k_1\right)}$$

$$= \frac{2\pi c}{\cos \pi k_0 \cos \pi k_1} \frac{2^{2k_1} n^{k_1-1/2} \Gamma\left(\frac{1}{2} + k_1\right)}{\Gamma\left(\frac{1}{2} + k_0 + k_1\right) \Gamma\left(\frac{1}{2} - k_0 + k_1\right)}.$$

$\qquad\square$

PROPOSITION 3. *Suppose* $-\frac{1}{2} < k_0 \pm k_1 < \frac{1}{2}$ *then*

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = 8 \int_0^{\pi/4} \phi\left(x_\theta\right)^{2n+1} p_{1,4}\left(x_\theta\right) K\left(x_\theta\right) p_{1,2}\left(x_\theta\right)^T d\theta$$

$$\sim \frac{-2\pi c}{\cos \pi k_0 \cos \pi k_1} \frac{2^{2k_1} n^{k_1 - 1/2} \Gamma\left(\frac{1}{2} + k_1\right)}{\Gamma\left(\frac{1}{2} + k_0 + k_1\right) \Gamma\left(\frac{1}{2} - k_0 + k_1\right)}.$$

PROOF. By definition

$$\phi\left(x_\theta\right)^{2n+1} p_{1,4}\left(x_\theta\right) K\left(x_\theta\right) p_{1,2}\left(x_\theta\right)^T = -\left(x_1^2 - x_2^2\right)^{2n+1}$$
$$\times \left\{ d_1\left(x_2 L_{11} + x_1 L_{12}\right)\left(x_2 L_{11} - x_1 L_{12}\right) + d_2\left(x_2 L_{21} + x_1 L_{22}\right)\left(x_2 L_{21} - x_1 L_{22}\right) \right\}.$$

Thus equations (2.3) and (2.4) imply

$$8 \int_0^{\pi/4} \phi\left(x_\theta\right)^{2n+1} p_{1,4}\left(x_\theta\right) K\left(x_\theta\right) p_{1,2}\left(x_\theta\right)^T d\theta$$
$$= -8 d_1 \frac{\left(1 - 2k_0 + 2k_1\right)\left(1 + 2k_0 + 2k_1\right)}{\left(1 + 2k_1\right)^2}$$
$$\times \int_0^1 \left(\frac{1 - u^2}{1 + u^2}\right)^{2n+1} \frac{u^{2k_1 + 2}}{\left(1 + u^2\right)^2} h_1\left(u^2\right) h_3\left(u^2\right) du$$
$$- 8 d_2 \int_0^1 \left(\frac{1 - u^2}{1 + u^2}\right)^{2n+1} \frac{u^{-2k_1}}{\left(1 + u^2\right)^2} h_2\left(u^2\right) h_4\left(u^2\right) du.$$

Arguing as in the previous proof, one obtains the same expressions (2.5) and (2.6) for the first and second integrals respectively.   □

3. Evaluation of the normalizing constant

The proof of the following appears in Section 4. (The Pochhammer symbol $(a)_n := \prod_{i=1}^n \left(a + i - 1\right)$.)

THEOREM 1. *For arbitrary* k_0, k_1 *and* $n \geq 0$

$$\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S = \frac{1}{n! \left(\frac{1}{2}\right)_{n+1}}$$
$$\times \sum_{j=0}^n \frac{(-n)_j^2}{j!} \left(-k_1\right)_j \left(\frac{1}{2} + k_1 + k_0\right)_{n+1-j} \left(\frac{1}{2} + k_1 - k_0\right)_{n-j},$$
$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = -\frac{1}{(n+1)! \left(\frac{1}{2}\right)_{n+1}}$$
$$\times \sum_{j=0}^n \frac{(-n)_j (-n-1)_j}{j!} \left(-k_1\right)_j \left(\frac{1}{2} + k_1 + k_0\right)_{n+1-j} \left(\frac{1}{2} + k_1 - k_0\right)_{n+1-j}.$$

COROLLARY 2. *Suppose* $-k_1 \pm k_0 \notin \frac{1}{2} + \mathbb{N}_0$ *then*

$$\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S = \frac{\left(\frac{1}{2} + k_1 + k_0\right)_{n+1} \left(\frac{1}{2} + k_1 - k_0\right)_n}{\left(\frac{1}{2}\right)_{n+1} n!}$$

$$\times \ {}_3F_2\left(\begin{matrix} -n, -n, -k_1 \\ -n - \frac{1}{2} - k_1 - k_0, -n + \frac{1}{2} - k_1 + k_0 \end{matrix}; 1\right),$$

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = -\frac{\left(\frac{1}{2} + k_1 + k_0\right)_{n+1} \left(\frac{1}{2} + k_1 - k_0\right)_{n+1}}{\left(\frac{1}{2}\right)_{n+1} (n+1)!}$$

$$\times \ {}_3F_2\left(\begin{matrix} -n, -n - 1, -k_1 \\ -n - \frac{1}{2} - k_1 - k_0, -n - \frac{1}{2} - k_1 + k_0 \end{matrix}; 1\right).$$

The next step is to compare the two hypergeometric series to ${}_2F_1\left(\begin{smallmatrix} -n, -k_1 \\ -n - 2k_1 \end{smallmatrix}; 1\right)$ which equals $\dfrac{(1 + k_1)_n}{(1 + 2k_1)_n}$ (Chu-Vandermonde sum). The following lemma will be used with $a = \frac{1}{2} + k_1 + k_0$, $b = -\frac{1}{2} + k_1 - k_0$, and $c = -k_1$.

LEMMA 3. *Suppose* $0 < a < 1$, $-1 < b < 0$ *and* $c > -1$ *then*

$${}_3F_2\left(\begin{matrix} -n, -n - 1, c \\ -n - a, -n - b - 1 \end{matrix}; 1\right) \le \frac{(1 + a + b + c)_n}{(1 + a + b)_n} \le {}_3F_2\left(\begin{matrix} -n, -n, c \\ -n - a, -n - b \end{matrix}; 1\right), c \ge 0;$$

$${}_3F_2\left(\begin{matrix} -n, -n, c \\ -n - a, -n - b \end{matrix}; 1\right) \le \frac{(1 + a + b + c)_n}{(1 + a + b)_n} \le {}_3F_2\left(\begin{matrix} -n, -n - 1, c \\ -n - a, -n - b - 1 \end{matrix}; 1\right), c \le 0.$$

PROOF. If $c = 0$ then each expression equals 1. The middle term equals ${}_2F_1\left(\begin{smallmatrix} -n, c \\ -n - a - b \end{smallmatrix}; 1\right)$. For $0 \le i \le n$ set

$$s_i := \frac{(-n)_i^2 (c)_i}{i! \, (-n - a)_i \, (-n - b)_i},$$

$$t_i := \frac{(-n)_i \, (c)_i}{i! \, (-n - a - b)_i},$$

$$u_i := \frac{(-n)_i \, (-n - 1)_i \, (c)_i}{i! \, (-n - a)_i \, (-n - b - 1)_i}.$$

Note $s_0 = t_0 = u_0 = 1$. From the relation $(-n - d)_i = (-1)^i (n - i + 1 + d)_i$ it follows that $\operatorname{sign}(s_i) = \operatorname{sign}(t_i) = \operatorname{sign}(u_i) = \operatorname{sign}((c)_i)$ for each i with $1 \le i \le n$ (by hypothesis $a + 1 > 2$ and $b + 1 > 0$). If $c > 0$ then $\operatorname{sign}((c)_i) = 1$ for all i and if $-1 < c < 0$ then $\operatorname{sign}((c)_i) = -1$ for $i \ge 1$. We find

$$\frac{s_i}{t_i} = \frac{(-n)_i \, (-n - a - b)_i}{(-n - a)_i \, (-n - b)_i} = \frac{(n - i + 1)(n - i + 1 + a + b)}{(n - i + 1 + a)(n - i + 1 + b)} \frac{s_{i-1}}{t_{i-1}}, i \ge 1,$$

and

$$\frac{m(m + a + b)}{(m + a)(m + b)} = 1 + \frac{-ab}{(m + a)(m + b)} > 1, m \ge 1,$$

because $-ab > 0$ and $b > -1$ (setting $m = n - i + 1$). This shows the sequence $\dfrac{s_i}{t_i}$ is positive and increasing. Also

$$\frac{u_i}{t_i} = \frac{(-n - 1)_i \, (-n - a - b)_i}{(-n - a)_i \, (-n - b - 1)_i} = \frac{(n - i + 2)(n - i + 1 + a + b)}{(n - i + 1 + a)(n - i + 2 + b)} \frac{u_{i-1}}{t_{i-1}}, i \ge 1,$$

$$\frac{(m+1)(m+a+b)}{(m+a)(m+b+1)} = 1 - \frac{b(a-1)}{(m+a)(m+b+1)} < 1, m \geq 1,$$

because $a < 1$ and $b < 0$ (and $a + b > -1$). The sequence $\frac{u_i}{t_i}$ is positive and decreasing. If $c > 0$ then $s_i, t_i, u_i > 0$ for $1 \leq i \leq n$ and $\frac{s_i}{t_i} > 1 > \frac{u_i}{t_i}$ implies $s_i > t_i > u_i$. This proves the first inequalities. If $-1 < c < 0$ then $s_i, t_i, u_i < 0$ for $1 \leq i \leq n$ and thus $\frac{s_i}{t_i} > 1 > \frac{u_i}{t_i}$ implies $s_i < t_i < u_i$. This proves the second inequalities. $\square$

Denote the $_3F_2$-sums in Corollary 2 by $f_1(n)$ and $f_2(n)$ respectively, then by using Stirling's formula

$$\frac{(a)_n}{(b)_n} = \frac{\Gamma(b)}{\Gamma(a)} \frac{\Gamma(a+n)}{\Gamma(b+n)} \sim \frac{\Gamma(b)}{\Gamma(a)} n^{a-b},$$

we find

$$\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S \sim \frac{\Gamma\left(\frac{1}{2}\right) f_1(n)}{\Gamma\left(\frac{1}{2} + k_1 + k_0\right) \Gamma\left(\frac{1}{2} + k_1 - k_0\right)} n^{2k_1 - \frac{1}{2}},$$

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S \sim - \frac{\Gamma\left(\frac{1}{2}\right) f_2(n)}{\Gamma\left(\frac{1}{2} + k_1 + k_0\right) \Gamma\left(\frac{1}{2} + k_1 - k_0\right)} n^{2k_1 - \frac{1}{2}}.$$

By Propositions 2 and 3 these imply for $i = 1, 2$

$$f_i(n) \sim \frac{2\pi c}{\cos \pi k_0 \cos \pi k_1} \times 2^{2k_1} n^{-k_1} \frac{\Gamma\left(\frac{1}{2} + k_1\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{2\pi c}{\cos \pi k_0 \cos \pi k_1} n^{-k_1} \frac{\Gamma(1 + 2k_1)}{\Gamma(1 + k_1)},$$

by the duplication formula. By Lemma 3 $f_1(n) \leq \frac{(1+k_1)_n}{(1+2k_1)_n} \leq f_2(n)$ for $0 \leq k_1 < \frac{1}{2}$, and the reverse inequality holds for $-\frac{1}{2} < k_1 < 0$. The fact that $\frac{(1+k_1)_n}{(1+2k_1)_n} \sim \frac{\Gamma(1+2k_1)}{\Gamma(1+k_1)} n^{-k_1}$ is the last ingredient for the proof of the following:

THEOREM 2. *Suppose* $-\frac{1}{2} < k_0 \pm k_1 < \frac{1}{2}$ *then the normalizing constant*

$$c = \frac{1}{2\pi} \cos \pi k_0 \cos \pi k_1.$$

The weight function K is integrable if the inequalities $-\frac{1}{2} < k_0, k_1 < \frac{1}{2}$ are satisfied (and the same value of c applies). However K is not positive-definite and integrable unless $-\frac{1}{2} < k_0 \pm k_1 < \frac{1}{2}$. It was shown in [1, p.21] that $\det K = d_1 d_2$. By using the (now-known) value of c and the values of d_1 and d_2 (see (1.2)) we find $\det K = d_1 d_2 = \frac{1}{4\pi^2} \cos \pi (k_0 + k_1) \cos \pi (k_0 - k_1)$.

4. Formulae for $\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S$ and $\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S$

The inner products are evaluated by computing $\Delta_\kappa^{2n}\left(\phi^{2n} p_{1,2}\right)$ and $\Delta_\kappa^{2n+1}\left(\phi^{2n+1} p_{1,4}\right)$ by means of recurrence relations. The start is a product formula for Δ_κ (using ∂_i to denote $\frac{\partial}{\partial x_i}$):

LEMMA 4. *Suppose $f(x)$ is a W-invariant polynomial and $g(x,t) \in \mathcal{P}_V$ then*

(4.1)

$$\Delta_\kappa(fg) - f\Delta_\kappa(g) = g\Delta f + 2\langle \nabla f, \nabla g\rangle$$

$$+ 2k_1\left(g(x,-t_1,t_2)\frac{\partial_1 f}{x_1} + g(x,t_1,-t_2)\frac{\partial_2 f}{x_2}\right)$$

$$+ 2k_0\left(g(x,t_2,t_1)\frac{\partial_1 f - \partial_2 f}{x_1 - x_2} + g(x,-t_2,-t_1)\frac{\partial_1 f + \partial_2 f}{x_1 + x_2}\right).$$

LEMMA 5. *Suppose $f \in \mathcal{P}_{V,2n+1}$ then $\Delta_\kappa^{n+1}|x|^2 f = 4(n+1)(n+2)\Delta_\kappa^n f$.*

PROOF. If $g \in \mathcal{P}_{V,m}$ then $\Delta_\kappa|x|^2 g = 4(m+1)g + |x|^2\Delta_\kappa g$ by [**1**, p.4, eq. (4)]. Apply Δ_κ repeatedly to this expression, and by induction one shows $\Delta_\kappa^\ell|x|^2 g = 4\ell(m-\ell+2)\Delta_\kappa^{\ell-1}g + |x|^2\Delta_\kappa^\ell g$ for $\ell = 1,2,3,\ldots$. Set $g = f, m = 2n+1$ and $\ell = n+1$ then $\Delta_\kappa^{n+1}f = 0$. $\qquad\square$

Recall that $\phi := x_1^2 - x_2^2$; and $\phi^{2n}p_{1,2}$ and $\phi^{2n+1}p_{1,4}$ are all relative invariants of the same type as $p_{1,2}$, that is, $\sigma_1 f = \sigma_{12}^+ f = -f$. Thus $\Delta_\kappa^{2n}(\phi^{2n}p_{1,2})$ and $\Delta_\kappa^{2n+1}(\phi^{2n+1}p_{1,4})$ are both scalar multiples of $p_{1,2}$, because Δ_κ commutes with the action of the group and $p_{1,2}$ is the unique degree-1 relative invariant of this type.

PROPOSITION 4. *For $n = 0,1,2,3,\ldots$*

(4.2)
$$\Delta_\kappa\phi^{2n}p_{1,2} = -8n(1+2k_1+2k_0)\phi^{2n-1}p_{1,4}$$
$$+ 8n(2n-1-2k_0)|x|^2\phi^{2n-2}p_{1,2},$$

(4.3)
$$\Delta_\kappa\phi^{2n+1}p_{1,4} = -4(2n+1)(1+2k_1-2k_0)\phi^{2n}p_{1,2}$$
$$+ 8n(2n+1+2k_0)|x|^2\phi^{2n-1}p_{1,4}.$$

PROOF. We use Lemma 4 with $f = \phi^{2n}$ and $g = p_{1,2}$ or $g = \phi p_{1,4}$. Simple computation shows that

$$\Delta\phi^{2n} = 8n(2n-1)|x|^2\phi^{2n-2}, \quad \nabla\phi^{2n} = 4n\phi^{2n-1}(x_1,-x_2),$$
$$\Delta_\kappa(\phi p_{1,4}) = -4(1+2k_1-2k_0)p_{1,2}.$$

For $g = p_{1,2}$ we find $2\langle\nabla\phi^{2n},\nabla g\rangle = -8n\phi^{2n-1}p_{1,4}$, the coefficient of k_1 in (4.1) is $-16n\phi^{2n-1}p_{1,4}$ and the coefficient of k_0 is $32n\phi^{2n-2}x_1x_2(x_1t_1 - x_2t_2)$. The first formula now follows from

$$x_1x_2(x_1t_1 - x_2t_2) = -\frac{1}{2}\left(\phi p_{1,4} + |x|^2 p_{1,2}\right).$$

For $g = \phi p_{1,4}$ we obtain $2\langle\nabla\phi^{2n},\nabla g\rangle = 8n\phi^{2n-1}\left(2|x|^2 p_{1,4} - \phi p_{1,2}\right)$. The coefficient of k_1 in (4.1) is $-16n\phi^{2n}p_{1,2}$ and the coefficient of k_0 is $-32n\phi^{2n-1}x_1x_2(x_1t_1 + x_2t_2)$. Similarly to the previous case

$$x_1x_2(x_1t_1 + x_2t_2) = -\frac{1}{2}\left(\phi p_{1,2} + |x|^2 p_{1,4}\right).$$

The proof of the second formula is completed by adding up the parts, including $\phi^{2n}\Delta_\kappa\phi p_{1,4}$. $\qquad\square$

We use this to set up a recurrence relation.

DEFINITION 2. *For $n = 0, 1, 2, \ldots$ the constants α_n, β_n implicitly depending on k_0, k_1 are defined by*

$$\left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S = \alpha_n \left\langle p_{1,2}, p_{1,2} \right\rangle_S = \alpha_n \left(1 + 2k_1 + 2k_0 \right),$$

$$\left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = \beta_n \left\langle p_{1,2}, p_{1,2} \right\rangle_S = \beta_n \left(1 + 2k_1 + 2k_0 \right).$$

Also $\alpha_n' := 2^{4n} (2n)! (2n+1)! \alpha_n$ and $\beta_n' := 2^{4n+2} (2n+1)! (2n+2)! \beta_n$.

PROPOSITION 5. *Suppose $n = 0, 1, 2, \ldots$ then $\Delta_\kappa^{2n} \left(\phi^{2n} p_{1,2} \right) = \alpha_n' p_{1,2}$, $\Delta_\kappa^{2n+1} \left(\phi^{2n+1} p_{1,4} \right) = \beta_n' p_{1,2}$ and α_n', β_n' satisfy the recurrence (with $\alpha_0' = 1, \beta_{-1}' = 0$)*

$$\beta_n' = -4 (2n+1) (1 + 2k_1 - 2k_0) \alpha_n' + 64n^2 (2n+1) (2n+1+2k_0) \beta_{n-1}',$$

$$\alpha_n' = -8n (1 + 2k_1 + 2k_0) \beta_{n-1}' + 64n^2 (2n-1) (2n-1-2k_0) \alpha_{n-1}', \ n \geq 1.$$

PROOF. By Proposition 1

$$2^{4n+1} (2n)! (2n+1)! \left\langle \phi^{2n} p_{1,2}, p_{1,2} \right\rangle_S = \left\langle \Delta_\kappa^{2n} \left(\phi^{2n} p_{1,2} \right), p_{1,2} \right\rangle_\tau$$

$$= \alpha_n \left\langle p_{1,2}, p_{1,2} \right\rangle_\tau = 2\alpha_n \left\langle p_{1,2}, p_{1,2} \right\rangle_S.$$

Similarly

$$2^{4n+3} (2n+1)! (2n+2)! \left\langle \phi^{2n+1} p_{1,4}, p_{1,2} \right\rangle_S = \left\langle \Delta_\kappa^{2n+1} \left(\phi^{2n+1} p_{1,4} \right), p_{1,2} \right\rangle_\tau$$

$$= \beta_n \left\langle p_{1,2}, p_{1,2} \right\rangle_\tau = 2\beta_n \left\langle p_{1,2}, p_{1,2} \right\rangle_S.$$

Apply Δ_κ^{2n-1} to both sides of equation (4.2) to obtain

$$\alpha_n' p_{1,2} = -8n (1 + 2k_1 + 2k_0) \beta_n' p_{1,2} + 8n (2n-1-2k_0) \Delta_\kappa^{2n-1} \left(|x|^2 \phi^{2n-2} p_{1,2} \right).$$

By Lemma 5

$$\Delta_\kappa^{2n-1} \left(|x|^2 \phi^{2n-2} p_{1,2} \right) = 8n (2n-1) \Delta_\kappa^{2n-2} \left(\phi^{2n-2} p_{1,2} \right) = 8n (2n-1) \alpha_{n-1}' p_{1,2}.$$

Apply Δ_κ^{2n} to both sides of equation (4.3) to obtain

$$\beta_n' p_{1,2} = -4 (2n+1) (1 + 2k_1 - 2k_0) \alpha_n' p_{1,2} + 8n (2n+1+2k_0) \Delta_\kappa^{2n} \left(|x|^2 \phi^{2n-1} p_{1,4} \right),$$

and by the same lemma

$$\Delta_\kappa^{2n} \left(|x|^2 \phi^{2n-1} p_{1,4} \right) = 8n (2n+1) \Delta_\kappa^{2n-1} \left(\phi^{2n-1} p_{1,4} \right) = 8n (2n+1) \beta_{n-1}' p_{1,2}.$$

$\square$

By a simple computation with the change of scale for α_n, β_n we obtain the following:

COROLLARY 3. *α_n, β_n satisfy the recurrence*

$$\beta_n = -\frac{1 + 2k_1 - 2k_0}{2 (n+1)} \alpha_n + \frac{n (2n+1+2k_0)}{(n+1) (2n+1)} \beta_{n-1},$$

$$\alpha_n = -\frac{1 + 2k_1 + 2k_0}{2n+1} \beta_{n-1} + \frac{2n-1-2k_0}{2n+1} \alpha_{n-1}.$$

The following formulae arose from examining values of α_n, β_n for some small n, obtained by using the recurrence and symbolic computation. In the calculations we use relations like $(a)_m (a+m) = (a)_{m+1}$ and $\frac{1}{(m-1)!} = \frac{m}{m!}$.

THEOREM 3. *Suppose $n = 0, 1, 2, \ldots$ then*

$$\alpha_n = \sum_{j=0}^{n} \frac{(-n)_j^2}{n! \left(\frac{3}{2}\right)_n j!} (-k_1)_j \left(\frac{3}{2} + k_1 + k_0\right)_{n-j} \left(\frac{1}{2} + k_1 - k_0\right)_{n-j},$$

$$\beta_n = -\sum_{j=0}^{n} \frac{(-n)_j (-1-n)_j}{(n+1)! \left(\frac{3}{2}\right)_n j!} (-k_1)_j \left(\frac{3}{2} + k_1 + k_0\right)_{n-j} \left(\frac{1}{2} + k_1 - k_0\right)_{n+1-j}.$$

For brevity $k_+ := k_1 + k_0$ and $k_- := k_1 - k_0$, as previously. We use induction on the sequence $\alpha_0 \to \beta_0 \to \alpha_1 \to \beta_1 \to \alpha_2 \to \cdots$. The formulae are clearly valid for $n = 0$. Suppose they are valid for some $n - 1$. Consider $-\frac{1+2k_1+2k_0}{2n+1} \beta_{n-1} + \frac{2n-1-2k_0}{2n+1} \alpha_{n-1}$; split up the j-term in α_{n-1} by writing $1 = \dfrac{j - k_1}{n - \frac{1}{2} - k_0} + \dfrac{n - \frac{1}{2} + k_1 - k_0 - j}{n - \frac{1}{2} - k_0}$, then

(4.4)

$$\frac{2n - 1 - 2k_0}{2n + 1} \alpha_{n-1} = \frac{1}{(n-1)! \left(\frac{3}{2}\right)_n} \sum_{j=0}^{n-1} \frac{1}{j!} (1 - n)_j^2 \left(\frac{3}{2} + k_+\right)_{n-1-j}$$

$$\times \left\{ (-k_1)_j \left(\frac{1}{2} + k_-\right)_{n-j} + (-k_1)_{j+1} \left(\frac{1}{2} + k_-\right)_{n-1-j} \right\},$$

(4.5)

$$-\frac{1 + 2k_1 + 2k_0}{2n + 1} \beta_{n-1} = \frac{\frac{1}{2} + k_+}{n! \left(\frac{3}{2}\right)_n} \sum_{j=0}^{n-1} \frac{1}{j!} (1 - n)_j (-n)_j (-k_1)_j \left(\frac{3}{2} + k_+\right)_{n-1-j} \left(\frac{1}{2} + k_-\right)_{n-j}.$$

The coefficient of $(-k_1)_j$ in the sum of the first part of { } in (4.4) and (4.5) is

$$\frac{1}{n! \left(\frac{3}{2}\right)_n j!} (1 - n)_j \left(\frac{3}{2} + k_+\right)_{n-1-j} \left(\frac{1}{2} + k_-\right)_{n-j} \left[n (1 - n)_j + \left(\frac{1}{2} + k_+\right)(-n)_j \right]$$

$$= \frac{1}{n! \left(\frac{3}{2}\right)_n j!} (1 - n)_j \left(\frac{3}{2} + k_+\right)_{n-1-j} \left(\frac{1}{2} + k_-\right)_{n-j} (-n)_j \left(n - j + \frac{1}{2} + k_+ \right)$$

$$= \frac{1}{n! \left(\frac{3}{2}\right)_n j!} (1 - n)_j \left(\frac{3}{2} + k_+\right)_{n-j} \left(\frac{1}{2} + k_-\right)_{n-j} (-n)_j.$$

For $j = 0$ this establishes the validity of the $j = 0$ term in α_n. For $1 \leq j \leq n$ replace j by $j - 1$ in the second part of { } in (4.4) and obtain

$$\frac{jn}{n! \left(\frac{3}{2}\right)_n j!} (1 - n)_{j-1}^2 \left(\frac{3}{2} + k_+\right)_{n-j} \left(\frac{1}{2} + k_-\right)_{n-j} (-k_1)_j;$$

adding all up leads to

$$\frac{1}{n! \left(\frac{3}{2}\right)_n j!} \left(\frac{3}{2} + k_+\right)_{n-j} \left(\frac{1}{2} + k_-\right)_{n-j} \left\{ (1 - n)_j (-n)_j + nj (1 - n)_{j-1}^2 \right\},$$

and the expression in { } evaluates to $(-n)_j^2$. This proves the validity of the formula for α_n.

To prove the formula for β_n consider $-\frac{1+2k_1-2k_0}{2(n+1)} \alpha_n + \frac{n(2n+1+2k_0)}{(n+1)(2n+1)} \beta_{n-1}$ and as before split up the j-term in β_{n-1} by writing $1 = \dfrac{j - k_1}{n + \frac{1}{2} + k_0} + \dfrac{n + \frac{1}{2} + k_1 + k_0 - j}{n + \frac{1}{2} + k_0}$.

Then
(4.6)
$$-\frac{1+2k_1-2k_0}{2(n+1)}\alpha_n = -\frac{\frac{1}{2}+k_1-k_0}{(n+1)!\left(\frac{3}{2}\right)_n}\sum_{j=0}^{n}\frac{1}{j!}(-n)_j^2(-k_1)_j\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n-j},$$
(4.7)
$$\frac{n(2n+1+2k_0)}{(n+1)(2n+1)}\beta_{n-1} = \frac{1}{(n+1)!\left(\frac{3}{2}\right)_n}\sum_{j=0}^{n-1}\frac{1}{j!}(-n)_{j+1}(-n)_j\left(\frac{1}{2}+k_-\right)_{n-j}$$
$$\times\left\{(-k_1)_j\left(\frac{3}{2}+k_+\right)_{n-j}+(-k_1)_{j+1}\left(\frac{3}{2}+k_+\right)_{n-j-1}\right\}.$$
The coefficient of $(-k_1)_j$ in the sum of (4.6) and the first part of{ } in (4.7) is
$$\frac{-1}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j^2\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n-j}\left(\frac{1}{2}+k_-+n-j\right)$$
$$=\frac{-1}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j^2\left(\frac{3}{2}+k_+\right)_{n-j}\left(\frac{1}{2}+k_-\right)_{n+1-j}.$$
For $j=0$ this establishes the validity of the $j=0$ term in β_n. For $1\le j\le n$ replace j by $j-1$ in the second part of { } in (4.7) and obtain
$$\frac{j}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j(-n)_{j-1}\left(\frac{1}{2}+k_-\right)_{n-j+1}\left(\frac{3}{2}+k_+\right)_{n-j};$$
adding all up leads to
$$\frac{-1}{(n+1)!\left(\frac{3}{2}\right)_n j!}(-n)_j\left(\frac{1}{2}+k_-\right)_{n-j+1}\left(\frac{3}{2}+k_+\right)_{n-j}\left\{(-n)_j-j(-n)_{j-1}\right\}$$
and the expression in { } evaluates to $(-1-n)_j$. This proves the validity of the formula for β_n and completes the induction.

This completes the proof of Theorem 1.

References

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